Des. Codes Cryptogr. (2013) 69:65–76
DOI 10.1007/s10623-012-9621-6

A new existence proof for Steiner quadruple systems

Xiande Zhang · Gennian Ge

Received: 16 March 2011 / Revised: 21 September 2011 / Accepted: 25 January 2012 /
Published online: 12 February 2012

Abstract A Steiner quadruple system of order v is an ordered pair $(X, \mathcal{B})$, where X is a set of cardinality v, and $\mathcal{B}$ is a set of 4-subsets of X, called blocks, with the property that every 3-subset of X is contained in a unique block. Such designs exist if and only if $v \equiv 2, 4 \pmod 6$. The first and second proofs of this result were given by Hanani in 1960 and in 1963, respectively. All the existing proofs are rather cumbersome, even though simplified proofs have been given by Lenz in 1985 and by Hartman in 1994. To study Steiner quadruple systems, Hanani introduced the concept of H-designs in 1963. The purpose of this paper is to provide an alternative existence proof for Steiner quadruple systems via H-designs of type 2^n. In 1990, Mills showed that for $n > 3$, $n \neq 5$, an H-design of type g^n exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by 3, where the main context is the complicated existence proof for H-designs of type 2^n. However, Mill's proof was based on the existence result of Steiner quadruple systems. In this paper, by using the theory of candelabra systems and H-frames, we give a new existence proof for H-designs of type 2^n independent of the existence result of Steiner quadruple systems. As a consequence, we also provide a new existence proof for Steiner quadruple systems.

Keywords Candelabra systems · H-designs · H-frames · Steiner quadruple systems

Mathematics Subject Classification (2000) 05B05

Communicated by K. T. Arasu.

X. Zhang · G. Ge (✉)
Department of Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang,
People's Republic of China
e-mail: gnge@zju.edu.cn

1 Introduction

A *Steiner quadruple system* of order v, denoted by SQS(v), is an ordered pair $(X, \mathcal{B})$, where X is a set of cardinality v, and $\mathcal{B}$ is a set of 4-subsets of X, called *blocks*, with the property that every 3-subset of X is contained in a unique block.

The necessary conditions for the existence of an SQS(v) are that $v \equiv 2, 4 \pmod 6$ or $v = 1$. When $v < 4$, the systems have no blocks, and when $v = 4$, it has one block. The smallest interesting system, SQS(8), was known to Kirkman [12] in 1847. The unique (up to isomorphism) SQS(10) was attributed to Barrau [1] as early as 1908 and to Richard Wilson in [3]. Several infinite families of quadruple systems were constructed by Kirkman [12] and by Carmichael [2]. The first complete proof for the existence of SQS(v) was given by Hanani [4] in 1960.

Theorem 1.1 *There exists an SQS(v) for all $v \equiv 2, 4 \pmod 6$.*

This result is proved by induction using six recursive constructions together with explicit constructions of an SQS(14) and an SQS(38). Hanani also gave a more sophisticated proof of the existence theorem for SQS(v) in [5], which relies on the construction of 3-wise balanced designs and 3-analogs of group divisible designs (the concept is defined below). Apart from Hanani's two proofs, Hartman [6–8] and Lenz [13] used the existence of candelabra quadruple systems (the concept is defined in Sect. 2) of type $(g^3 : s)$ with $s \in \{1, 2, 4, 8\}$ to give a purely tripling existence proof, which used only one type of construction and a small number of initial designs: SQS(v) with $v \in \{8, 10, 14\}$ and HQS$(v : 8)$ with $v \in \{26, 28, 32, 34, 38, 40\}$. For more information on Steiner quadruple systems, see the excellent survey paper by Hartman and Phelps [10].

Let K be a set of positive integers. A *group divisible* 3-*design* of order v with block sizes from K, denoted by GDD$(3, K, v)$, is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

(1) X is a set of v elements (called *points*);
(2) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of nonempty subsets (called *groups*) of X which partition X;
(3) $\mathcal{B}$ is a family of subsets (called *blocks*) of X, each of cardinality from K such that each block intersects any given group in at most one point;
(4) every 3-subset T of X from three distinct groups is contained in a unique block.

The *type* of the GDD$(3, K, v)$ is defined as the list $(|G||G \in \mathcal{G})$. If a GDD has n_i groups of size g_i, $1 \le i \le r$, then we use an "*exponential*" notation $g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r}$ to denote the group type. When $K = \{k\}$, we simply write k for K. A GDD is called *uniform* if all groups have the same size. Mills used H$(n, g, 4, 3)$ design to denote the GDD$(3, 4, ng)$ of type g^n. In this paper, we use H$(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r})$ to denote the GDD$(3, 4, \sum n_i g_i)$ of type $g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r}$ for short.

For the existence of uniform H-designs, Mills [15] showed that for $n > 3$, $n \neq 5$, an H(g^n) exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by 3, and that for $n = 5$, an H(g^5) exists if g is divisible by 4 or 6. Recently, Ji [11] improved these results by showing that an H(g^5) exists whenever g is even, $g \neq 2$ and $g \not\equiv 10, 26 \pmod{48}$. We summarize their results as follows.

Theorem 1.2 ([11,15]) *For $n > 3$ and $n \neq 5$, an H(g^n) exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by* 3. *For $n = 5$, an H(g^n) exists when g is even, $g \neq 2$ and $g \not\equiv 10, 26 \pmod{48}$.*

It is easy to see that the existence of an H(2^n) implies that of an SQS$(2n)$ by combining every two groups of the H(2^n) to form a quadruple as a new block. However, the existing proof for the existence of $\mathrm{H}(2^n)$, which is the main content of Mills' paper [15], is based on the existence result of Steiner quadruple systems. The purpose of this paper is to provide an alternative existence proof for Steiner quadruple systems via H-designs of type 2^n. By using the theory of candelabra systems and H-frames, we give a new existence proof for H-designs of type 2^n independent of the existence result of Steiner quadruple systems. As a consequence, we also provide a new existence proof for Steiner quadruple systems.

2 Definitions and recursive constructions

In this section, we shall describe several recursive constructions for H-designs from candelabra systems and H-frames.

A *candelabra t-system* (or t-CS) of order v and block sizes from K, denoted by $\mathrm{CS}(t, K, v)$, is a quadruple $(X, S, \Gamma, \mathcal{A})$ that satisfies the following properties:

(1) X is a set of v elements (called *points*);
(2) S is an s-subset (called the *stem* of the *candelabra*) of X;
(3) $\Gamma = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets (called *groups* or *branches*) of $X \backslash S$, which partition $X \backslash S$;
(4) $\mathcal{A}$ is a collection of subsets (called *blocks*) of X, each of cardinality from K;
(5) every t-subset T of X with $|T \cap (S \cup G_i)| < t$, for all i, is contained in a unique block of $\mathcal{A}$, and no t-subset of $S \cup G_i$, for any i, is contained in any block of $\mathcal{A}$.

By the *group type* of a t-CS $(X, S, \Gamma, \mathcal{A})$ we mean the list $(|G| | G \in \Gamma : |S|)$ of group sizes and stem size. If a t-CS has n_i groups of size g_i, $1 \le i \le r$ and stem size s, then we use the notation $(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ to denote the group type. Such a candelabra system will be denoted by t-$\mathrm{CS}(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$. A candelabra system with $t = 3$ and $K = \{4\}$ is called a *candelabra quadruple system* and denoted by $\mathrm{CQS}(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$.

A $\mathrm{CS}(t, K, v)$ of type $(1^v : 0)$ $(X, S, \Gamma, \mathcal{A})$ is usually called a *t-wise balanced design* and briefly denoted by $\mathrm{S}(t, K, v)$. The stem and the group set are often omitted and we write a pair $(X, \mathcal{A})$ instead of a quadruple $(X, S, \Gamma, \mathcal{A})$. It is well known that an $S(3, \{4, 6\}, v)$ exists if and only if $v \equiv 0 \pmod 2$ [5].

The following is a construction for 3-CSs which is a special case of the fundamental construction of Hartman [8].

Theorem 2.1 *Suppose that $(X, \mathcal{A})$ is an $S(t, K', v)$ and $\infty \in X$. Let $K_1 = \{|A| : \infty \in A \in \mathcal{A}\}$ and $K_2 = \{|A| : \infty \notin A \in \mathcal{A}\}$. If there exists a $CS(3, K, t(k_1 - 1) + a)$ of type $(t^{k_1 - 1} : a)$ for each $k_1 \in K_1$ and a $GDD(3, K, tk_2)$ of type t^{k_2} for each $k_2 \in K_2$, then there exists a $CS(3, K, t(v - 1) + a)$ of type $(t^{v-1} : a)$.*

For non-negative integers q, g, k and t, an *$H(q, g, k, t)$ frame* (as in [9]) is an ordered four-tuple $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with the following properties:

1. X is a set of qg points;
2. $\mathcal{G} = \{G_1, G_2, \ldots, G_q\}$ is an equipartition of X into q groups;
3. $\mathcal{F}$ is a family $\{F_i\}$ of subsets of $\mathcal{G}$ called *holes*, which is closed under intersections. Hence each hole $F_i \in \mathcal{F}$ is of the form $F_i = \{G_{i_1}, G_{i_2}, \ldots, G_{i_s}\}$, and if F_i and F_j are holes then $F_i \cap F_j$ is also a hole. The number of groups in a hole is its *size*; and
4. $\mathcal{B}$ is a set of k-element transverses (called *blocks*) of $\mathcal{G}$ with the property that every t-element transverse of $\mathcal{G}$, which is not a t-element transverse of any hole $F_i \in \mathcal{F}$ is contained in precisely one block, and no block contains a t-element transverse of any hole, where a *transverse* is a subset of X that meets each G_i in at most one point.

In this paper, an H(q, g, k, t) frame is shortly denoted by HF(q, g, k, t). If an HF$(q, g, 4, 3)$ has n_i holes of size $m_i + s$ intersecting on a common hole of size s, $i = 1, 2, \ldots, r$, then we denote such a design as HF$_g(m_1^{n_1} m_2^{n_2} \ldots m_r^{n_r} : s)$. It is clear that an HF$_1(m_1^{n_1} m_2^{n_2} \ldots m_r^{n_r} : s)$ is just a CQS$(m_1^{n_1} m_2^{n_2} \ldots m_r^{n_r} : s)$. If an HF$(q, g, 4, 3)$ has only one hole of size s, then we call it an *incomplete H-design* of type $(g^q : g^s)$, denoted by IH$(g^q : g^s)$.

Lemma 2.2 *Suppose that* $(X, S, \Gamma, \mathcal{A})$ *is a* 3*-CS*$(m_1^{n_1} m_2^{n_2} \ldots m_r^{n_r} : s)$ *and* $\infty \in S$. *Let* $K_1 = \{|A| : \infty \in A \in \mathcal{A}\}$ *and* $K_2 = \{|A| : \infty \notin A \in \mathcal{A}\}$. *If there exists an HF*$_g(t^{k_1 - 1} : a)$ *for each* $k_1 \in K_1$ *and an H*$((gt)^{k_2})$ *for each* $k_2 \in K_2$, *then there exists an HF*$_g((tm_1)^{n_1} (tm_2)^{n_2} \ldots (tm_r)^{n_r} : t(s-1)+a)$. *Furthermore, if* $4 \in K_2$, *then the resulting HF*$_g((tm_1)^{n_1} (tm_2)^{n_2} \ldots (tm_r)^{n_r} : t(s-1)+a)$ *contains a subdesign H*(g^4).

Proof Suppose $(X, S, \Gamma, \mathcal{A})$ is the given 3-CS$(m_1^{n_1} m_2^{n_2} \ldots m_r^{n_r} : s)$ with group set $\Gamma = \{G_1, \ldots, G_n\}$, where $n = \sum_{i=1}^r n_i$. Define $G'_{x,j} = \{x\} \times \{j\} \times Z_g$. Let $X' = ((X \backslash \{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times Z_a \times Z_g)$, $\mathcal{G}' = \{G'_{x,j} : x \in X \setminus \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$, $\mathcal{F} = \{F_i : 0 \le i \le n\}$, where $F_0 = \{G'_{x,j} : x \in S \backslash \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$ and $F_i = \{G'_{x,j} : x \in G_i, j \in Z_t\} \cup F_0$ for $1 \le i \le n$.

For each $B \in \mathcal{A}$ and $\infty \in B$, construct an HF$_g(t^{|B|-1} : a)$ on $((B \backslash \{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times Z_a \times Z_g)$ with group set $\{G'_{x,j} : x \in B \setminus \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$ and hole set $\mathcal{F}_B = \{F_x : x \in B\}$, where $F_x = \{G'_{x,j} : j \in Z_t\} \cup F_\infty$ with $F_\infty = \{G'_{\infty,j} : j \in Z_a\}$ being the common hole of size a. Denote its block set by $\mathcal{C}_B$.

For each $B \in \mathcal{A}$ and $\infty \notin B$, construct an H$((gt)^{|B|})$ on $B \times Z_t \times Z_g$ with group set $\{\{x\} \times Z_t \times Z_g : x \in B\}$. Denote its block set by $\mathcal{D}_B$.

Let $\mathcal{A}' = (\bigcup_{\infty \in B, B \in \mathcal{A}} \mathcal{C}_B) \bigcup (\bigcup_{\infty \notin B, B \in \mathcal{A}} \mathcal{D}_B)$. It is easy to check that $(X', \mathcal{G}', \mathcal{A}', \mathcal{F})$ forms an HF$_g((tm_1)^{n_1} (tm_2)^{n_2} \ldots (tm_r)^{n_r} : t(s-1)+a)$ with F_0 being the common hole of size $t(s-1)+a$.

Furthermore, if $4 \in K_2$, then there exists a block $B_0 = \{a, b, c, d\} \in \mathcal{A}$ and $\infty \notin B_0$. Now, we construct an H$((gt)^4)$ on $B_0 \times Z_t \times Z_g$ with group set $\mathcal{G}'_{B_0} = \{\{x\} \times Z_t \times Z_g : x \in B_0\}$ as follows. First, we construct an H(t^4) on $B_0 \times Z_t$ with group set $\{\{x\} \times Z_t : x \in B_0\}$ and block set $\mathcal{E}$. Next, for each $E = \{(a, i), (b, j), (c, k), (d, l)\} \in \mathcal{E}$, construct an H$(g^4)$ on $E \times Z_g$ with group set $\{\mathcal{G}'_{a,i}, \mathcal{G}'_{b,j}, \mathcal{G}'_{c,k}, \mathcal{G}'_{d,l}\}$ and block set $\mathcal{U}_E$. Then $\mathcal{D}_{B_0} = \bigcup_{E \in \mathcal{E}} \mathcal{U}_E$ is the block set of an H$((gt)^4)$ on $B_0 \times Z_t \times Z_g$ with group set $\mathcal{G}'_{B_0}$. Thus, each $\mathcal{U}_E$ forms the block set of a subdesign H(g^4) of the resulting HF$_g((tm_1)^{n_1} (tm_2)^{n_2} \ldots (tm_r)^{n_r} : t(s-1)+a)$. □

The following two constructions are modifications of the filling holes construction for Steiner quadruple systems using candelabra quadruple systems.

Lemma 2.3 *Suppose that there exists an HF*$_g(m_0^1 m_1^{n_1} m_2^{n_2} \ldots m_r^{n_r} : s)$. *Let* $n = m_0 + \sum_{i=1}^r m_i n_i + s$.

(1) *If there exists an IH*$(g^{m_i+s} : g^s)$ *for each* $i = 1, 2, \ldots, r$, *then there exists an IH*$(g^n : g^{m_0+s})$. *Furthermore, if there is an H*(g^{m_0+s}), *then there is an H*(g^n).
(2) *Let* $\epsilon = 0$ *or* 1. *If there exists an H*$(g^{m_i+\epsilon}(gs - g\epsilon)^1)$ *for each* $i = 0, 1, 2, \ldots, r$, *then there exists an H*$(g^{n-s+\epsilon}(gs - g\epsilon)^1)$.

Proof The proof of (1) is obvious. We only give the proof for (2). Let $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ be the given HF$_g(m_0^1 m_1^{n_1} m_2^{n_2} \ldots m_r^{n_r} : s)$. Let $F_0 = \{G_{\infty,1}, G_{\infty,2}, \ldots, G_{\infty,s}\}$ be the common hole. When $\epsilon = 0$, for each hole $F = \{G_1, G_2, \ldots, G_{m_i}\} \cup F_0$ of size $m_i + s$ with $i \in$

$\{0, 1, 2, \ldots, r\}$, construct an $\mathrm{H}(g^{m_i}(gs)^1)$ on $\cup_{G\in F} G$ with group set $\{G_1, G_2, \ldots, G_{m_i}\} \cup \{\cup_{G\in F_0} G\}$ and block set $\mathcal{A}_F$. Then $\mathcal{B} \cup (\cup_{F\in\mathcal{F}\setminus\{F_0\}}\mathcal{A}_F)$ is the block set of an $\mathrm{H}(g^{n-s}(gs)^1)$ with group set $\{G \in F\setminus F_0 : F \in \mathcal{F}\} \cup \{\cup_{G\in F_0} G\}$. When $\epsilon = 1$, for each hole $F = \{G_1, G_2, \ldots, G_{m_i}\} \cup F_0$ of size $m_i + s$ with $i \in \{0, 1, 2, \ldots, r\}$, construct an $\mathrm{H}(g^{m_i+1}(gs - g)^1)$ on $\cup_{G\in F} G$ with group set $\{G_1, G_2, \ldots, G_{m_i}, G_{\infty,1}\} \cup \{(\cup_{G\in F_0} G)\setminus G_{\infty,1}\}$ and block set $\mathcal{C}_F$. Then $\mathcal{B} \cup (\cup_{F\in\mathcal{F}\setminus\{F_0\}}\mathcal{C}_F)$ is the block set of an $\mathrm{H}(g^{n-s+1}(gs - g)^1)$ with group set $\{G \in F\setminus F_0 : F \in \mathcal{F}\} \cup \{G_{\infty,1}\} \cup \{(\cup_{G\in F_0} G)\setminus G_{\infty,1}\}$. □

Now we give two tripling constructions and a doubling construction for $\mathrm{H}(2^n)$. The two tripling constructions are variations of those for $\mathrm{SQS}(v)$ proposed by Hartman in [6] and [7], which will play a similar role to that of the tripling constructions of Hartman [6–8] and Lenz [13] to deal with $\mathrm{SQS}(v)$. First, we need the following definitions and notations.

A regular graph (V, E) of degree k is said to have a *one-factorization* if the edge set E can be partitioned into k parts $E = F_1|F_2|\ldots|F_k$ so that each F_i is a partition of the vertex set V into pairs. The parts F_i are called *one-factors*.

For $x \in Z_n$, we define $|x|$ by x if $0 \le x \le n/2$ and $n - x$ if $n/2 < x < n$. For $n \ge 2$ and $L \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, define $\mathrm{G}(n, L)$ to be the regular graph with vertex set Z_n and edge set E given by $\{x, y\} \in E$ if and only if $|x - y| \in L$.

The following lemma was proved by Stern and Lenz in [16].

Lemma 2.4 *Let $L \subseteq \{1, 2, \ldots, n\}$. Then $G(2n, L)$ has a one-factorization if and only if $2n/gcd(j, 2n)$ is even for some $j \in L$.*

For non-negative integers n and $s \ge 1$, a *simple pairing $P(n, 2s)$* (as in [6]) consists of four subsets Δ, R_0, R_1, R_2 of Z_{6n+2s} and three subsets PR_0, PR_1, PR_2 of $Z_{6n+2s} \times Z_{6n+2s}$ with the following properties for each $i \in \{0, 1, 2\}$:

(1) *Cardinality and symmetry conditions*

  (a) $|\Delta| = 2s$, $|R_i| = 2n$,
  (b) $\Delta = -\Delta$.

(2) *Partitioning conditions*

  (a) PR_i is a partition of R_i into pairs, thus $|PR_i| = n$,
  (b) Δ, R_0, R_1, R_2 is a partition of the set Z_{6n+2s}, i.e., $Z_{6n+2s} = \Delta \cup R_0 \cup R_1 \cup R_2$.

(3) *Pairing conditions*
Let $L_i = \{|x - y| : \{x, y\} \in PR_i\}$,

  (a) $3n + s \notin L_i$,
  (b) $|L_i| = n$,
  (c) $G_i = G(6n + 2s, \{1, 2, \ldots, 3n + s\}\setminus L_i)$ has a one-factorization.

Theorem 2.5 *For each pair of integers $n \ge 0$ and $s \ge 1$, there exists a simple pairing $P(n, 2s)$ with the extra property that $\{0, 3n + s\} \subset \Delta$ and G_i has a one-factorization with $\{\{k, k + 3n + s\} : 0 \le k \le 3n + s - 1\}$ as one of the one-factors for each $i \in \{0, 1, 2\}$.*

Proof For each pair of integers $n \ge 0$ and $s \ge 1$, a $P(n, 2s)$ was constructed in [6, Theorem 3.3]. It is easy to check that $\{0, 3n + s\} \subset \Delta$. The lengths L_i of all $P(n, 2s)$s for each $i \in \{0, 1, 2\}$ are listed below:

Case (a) $s = 1$ and n even, or $s \ge 2$.

$L_0 = \{2j : 0 < j \le \lfloor n/2 \rfloor \text{ or } n < j \le n + \lceil n/2 \rceil\}$,

$L_1 = \{2j : \lfloor n/2 \rfloor < j \le n + \lfloor n/2 \rfloor\}$,
$L_2 = \{2j : 0 < j \le n\}$.

Case (b) $n = 2k + 1$, $k \ge 0$ and $s = 1$.

$L_0 = \{2j : 0 < j \le k, 2k < j \le 3k + 1\}$,
$L_1 = \{2j : k < j \le 3k\} \cup \{1\}$,
$L_2 = \{2j : 0 < j \le 2k\} \cup \{1\}$.

Let $G'_i = G(6n + 2s, \{1, 2, \dots, 3n + s\} \backslash (L_i \cup \{3n + s\}))$, $i \in \{0, 1, 2\}$. By Lemma 2.4, each of G'_i and $G(6n + 2s, \{3n + s\})$ has a one-factorization. Hence, G_i has a one-factorization with $\{\{k, k + 3n + s\} : 0 \le k \le 3n + s - 1\}$ as one of the one-factors for each $i \in \{0, 1, 2\}$. □

Example 1 [6] Let $n = 1$ and $s = 1$. Construct a $P(1, 2)$ on Z_8 as follows:

$$\Delta = \{0, 4\},\ PR_0 = \{\{3, 5\}\},\ PR_1 = \{\{1, 2\}\},\ PR_2 = \{\{6, 7\}\}.$$

Note that each of the graphs $G_0 = G(8, \{1, 3, 4\})$, $G_1 = G(8, \{2, 3, 4\})$ and $G_2 = G(8, \{2, 3, 4\})$ has a one-factorization with $\{\{k, k + 4\} : 0 \le k \le 3\}$ as one of the one-factors.

Theorem 2.6 *There exists an* $HF_2((3n + s)^3 : s)$ *with a subdesign* $H(2^4)$ *for each pair of integers* $n \ge 0$ *and* $s \ge 1$.

Proof By Theorem 2.5, for each pair of integers $n \ge 0$ and $s \ge 1$, there is a simple pairing $P(n, 2s)$: Δ, R_i, PR_i, such that $\{0, 3n + s\} \subset \Delta$ and G_i has a one-factorization $F_i^{(1)}|F_i^{(2)}|\dots|F_i^{(4n+2s-1)}$ with $F_i^{(1)} = \{\{k, k + 3n + s\} : 0 \le k \le 3n + s - 1\}$ for each $i \in \{0, 1, 2\}$. Using this simple pairing, Hartman [6, Theorem 3.4] constructed a $\mathrm{CQS}((6n + 2s)^3 : 2s)$ on the point set $X = \{a_i : a \in Z_{6n+2s}, i \in \{0, 1, 2\}\} \cup \{\infty_1, \infty_2, \dots, \infty_{2s}\}$ with three groups $\{\{a_i : a \in Z_{6n+2s}\} : i \in \{0, 1, 2\}\}$ and a stem $\{\infty_1, \infty_2, \dots, \infty_{2s}\}$, as well as the block set $\mathcal{B}$ consisting of the following three parts:

$$\begin{aligned}
\delta &= \{\{\infty_j, (a + d)_0, (b - d)_1, (c + d)_2\} : a + b + c \equiv 0 \pmod{6n + 2s},\\
&\qquad d \text{ is the } j\text{th member of } \Delta, 1 \le j \le 2s\},\\
\rho &= \{\{(a + q)_i, (a + t)_i, b_{i+1}, c_{i+2}\} : a + b + c \equiv 0 \pmod{6n + 2s},\\
&\qquad \{q, t\} \in PR_i, i \in Z_3\}, \text{ and}\\
\phi &= \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a, b\} \in F_i^{(k)}, \{c, d\} \in F_{i+1}^{(k)}, 1 \le k \le 4n + 2s - 1, i \in Z_3\}.
\end{aligned}$$

Let

$$\phi_1 = \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a, b\} \in F_i^{(1)}, \{c, d\} \in F_{i+1}^{(1)}, i \in Z_3\}.$$

The desired $\mathrm{HF}_2((3n+s)^3 : s)$ will be on X with the group set $\mathcal{G} = \{\{k_i, (k+3n+s)_i\} : 0 \le k \le 3n+s-1, i \in \{0, 1, 2\}\} \cup \{\{\infty_i, \infty_{i+s}\} : 1 \le i \le s\}$, three holes $\{\{k_i, (k+3n+s)_i\} : 0 \le k \le 3n+s-1\} \cup \mathcal{F}_0$, $i \in \{0, 1, 2\}$ and a common hole $\mathcal{F}_0 = \{\{\infty_i, \infty_{i+s}\} : 1 \le i \le s\}$, as well as the block set $\mathcal{B} \backslash \phi_1$.

Since $\{0, 3n + s\} \subset \Delta$, without loss of generality we may assume 0, $3n + s$ are the first and the $(s + 1)$th elements of Δ respectively. Let

$$\begin{aligned}
\delta_0 = \{\{\infty_j, (a + d)_0, (b - d)_1, (c + d)_2\} &: a + b + c \equiv 0 \pmod{6n + 2s},\\
&a, b, c \in \{0, 3n + s\}, d \text{ is the } j\text{th member of } \Delta \text{ and } j = 1 \text{ or } s + 1\}.
\end{aligned}$$

Note that $\delta_0 \subset \delta$ and δ_0 forms the block set of an H(2^4) with the group set $\{\{0_i, (3n+s)_i\} : i \in \{0, 1, 2\}\} \cup \{\{\infty_1, \infty_{1+s}\}\}$. Hence, the above HF$_2((3n+s)^3 : s)$ contains a subdesign H(2^4). □

Example 1 (continued): Using the foregoing $P(1, 2)$, we may construct a CQS$(8^3 : 2)$ on the point set $X = \{a_i : a \in Z_8, i \in \{0, 1, 2\}\} \cup \{\infty_1, \infty_2\}$ with three groups $\{\{a_i : a \in Z_8\} : i \in \{0, 1, 2\}\}$ and a stem $\{\infty_1, \infty_2\}$, as well as the block set $\mathcal{B}$ consisting of the following three sets:

$$\begin{aligned}
\delta &= \{\{\infty_1, a_0, b_1, c_2\}, \{\infty_2, (a+4)_0, (b-4)_1, (c+4)_2\} : a+b+c \equiv 0 \pmod 8\},\\
\rho &= \{\{(a+3)_0, (a+5)_0, b_1, c_2\}, \{(a+1)_1, (a+2)_1, b_2, c_0\},\\
&\qquad \{(a+6)_2, (a+7)_2, b_0, c_1\} : a+b+c \equiv 0 \pmod 8\}, \text{ and}\\
\phi &= \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a, b\} \in F_i^{(k)}, \{c, d\} \in F_{i+1}^{(k)}, 1 \le k \le 5, i \in Z_3\}.
\end{aligned}$$

Here, $F_i^{(1)} | F_i^{(2)} | \ldots | F_i^{(5)}$ is a one-factorization of G_i with $F_i^{(1)} = \{\{k, k+4\} : 0 \le k \le 3\}$ for each $i \in \{0, 1, 2\}$. Let $\phi_1 = \{\{k_i, (k+4)_i, k'_{i+1}, (k'+4)_{i+1}\} : 0 \le k, k' \le 3, i \in Z_3\} \subset \phi$. The block set $(\delta \cup \rho \cup \phi) \backslash \phi_1$ forms an HF$_2(4^3 : 1)$ on X with the group set $\{\{k_i, (k+4)_i\} : 0 \le k \le 3, i \in \{0, 1, 2\}\} \cup \{\{\infty_1, \infty_2\}\}$, three holes $\{\{k_i, (k+4)_i\} : 0 \le k \le 3\} \cup \mathcal{F}_0, i \in \{0, 1, 2\}$ and a common hole $\mathcal{F}_0 = \{\{\infty_1, \infty_2\}\}$. Furthermore, as a subset of δ, $\delta_0 = \{\{\infty_1, a_0, b_1, c_2\}, \{\infty_2, (a+4)_0, (b-4)_1, (c+4)_2\} : a, b, c \in \{0, 4\}, a+b+c \equiv 0 \pmod 8\}$ forms an H(2^4) with group set $\{\{0_i, 4_i\} : i \in \{0, 1, 2\}\} \cup \{\{\infty_1, \infty_2\}\}$.

As a consequence of Theorem 2.6, we have our first tripling construction as follows.

Corollary 2.7 (Tripling Construction I) *Let* $n \equiv 2s \pmod 3$ *and* $s \ge 1$. *If there exists an IH*$(2^n : 2^s)$, *then there exists an IH*$(2^{3n-2s} : 2^n)$ *and an IH*$(2^{3n-2s} : 2^s)$. *Furthermore, if there exists an H*(2^n), *then there exists an IH*$(2^{3n-2s} : 2^4)$ *and an H*(2^{3n-2s}).

Proof By Theorem 2.6, we have an HF$_2((n-s)^3 : s)$ with a subdesign H(2^4). Filling in the first two holes with an IH$(2^n : 2^s)$, we obtain an IH$(2^{3n-2s} : 2^n)$ with a subdesign H(2^4). Filling in an IH$(2^n : 2^s)$ to this resultant IH$(2^{3n-2s} : 2^n)$, we obtain an IH$(2^{3n-2s} : 2^s)$. Filling in an H(2^n) instead, we obtain an H(2^{3n-2s}) with a subdesign H(2^4), which is also an IH$(2^{3n-2s} : 2^4)$. □

Theorem 2.8 *There exists an HF*$_2((3n)^3 : s)$ *for each pair of integers* n, s *such that* $3n \ge s \ge 0$.

Proof For each pair of integers n, s such that $3n \ge s \ge 0$ and $(n, s) \ne (1, 1)$, the proof is similar to that of Theorem 2.6. We may start from a particular CQS$((6n)^3 : 2s)$ and partition the points of each group into disjoint pairs. Then, we can remove the blocks formed by all the pairs from different groups. Such a CQS$((6n)^3 : 2s)$ was constructed by Hartman in [7, Sect. 4] on $X = \{a_i : a \in Z_{6n}, i \in \{0, 1, 2\}\} \cup \{\infty_1, \infty_2, \ldots, \infty_{2s}\}$ with three groups $\{\{a_i : a \in Z_{6n}\} : i \in \{0, 1, 2\}\}$ and stem $\{\infty_1, \infty_2, \ldots, \infty_{2s}\}$, as well as the block set $\mathcal{B}$ containing the following blocks:

$$\phi = \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a, b\} \in F_i^{(k)}, \{c, d\} \in F_{i+1}^{(k)}, 1 \le k \le 6n-1-2r-2h, i \in Z_3\},$$

where $F_i^{(1)}, F_i^{(2)}, \ldots, F_i^{(6n-1-2r-2h)}$ are disjoint partitions of pairs of Z_{6n} for each $i \in \{0, 1, 2\}$ and r, h are non-negative integers such that $6n = 2s + 2h + 6r$.

An $\mathrm{HF}_2(3^3 : 1)$ can be constructed by applying Lemma 2.2 with a $\mathrm{CQS}(3^3 : 1)$ in [4] and an $\mathrm{H}(2^4)$. □

As a consequence of Theorem 2.8, we have our second tripling construction as follows.

Corollary 2.9 (Tripling Construction II) *Let* $n \equiv s \pmod 3$ *and* $s \geq 0$. *If there exists an* $IH(2^n : 2^s)$, *then there exists an* $IH(2^{3n-2s} : 2^n)$ *and an* $IH(2^{3n-2s} : 2^s)$.

Theorem 2.10 (Doubling Construction) *If there exists an* $H(2^n)$, *then there exists an* $H(2^{2n})$.

Proof Let $(X, \mathcal{G}, \mathcal{B})$ be the given $\mathrm{H}(2^n)$. Let $\mathcal{F} = \{F_1, \ldots, F_{2(n-1)}\}$ be a one-factorization of the multi-partite complete graph on X with partite set $\mathcal{G}$. The desired $\mathrm{H}(2^{2n})$ is based on $X \times \{0, 1\}$ with $2n$ groups $G \times \{i\}$, $G \in \mathcal{G}$ and $i \in \{0, 1\}$. The block set is $\mathcal{A} = (\mathcal{B} \times \{0, 1\}) \cup \mathcal{C}$, where $\mathcal{C} = \{\{(a, 0), (b, 0), (c, 1), (d, 1)\} : \{a, b\}, \{c, d\} \in F_i, 1 \leq i \leq 2(n-1)\}$. □

3 An alternative existence proof for $\mathbf{H(2^n)}$

In this section, we give an alternative existence proof for $\mathrm{H}(2^n)$ with $n \equiv 1, 2 \pmod 3$ and $n \neq 5$, which is mainly based on the recursive constructions listed in Sect. 2. The proof is independent of the existence result of Steiner quadruple systems. Hence, we also give a new proof for the existence of $\mathrm{SQS}(v)$ in the meantime. First, we need the following initial ingredient designs.

Lemma 3.1 [5,14,15] *There exists an* $H(2^k)$ *for each* $k \in \{7, 11, 13\}$, *an* $H(6^k)$ *for each* $k \in \{4, 6\}$ *and an* $IH(2^{11} : 2^5)$.

Proof An $\mathrm{H}(2^7)$ can be found in [5]. An $\mathrm{H}(2^{11})$, an $\mathrm{H}(2^{13})$ and an $\mathrm{IH}(2^{11} : 2^5)$ were constructed by Mills in [15]. An $\mathrm{H}(6^k)$ for each $k \in \{4, 6\}$ exists by [14, Lemma 7]. □

Lemma 3.2 *There exists an* $H(2^{25})$.

Proof We will construct an $\mathrm{H}(2^{25})$ on $X = Z_{25} \times Z_2$ with the group set $\mathcal{G} = \{G_i = \{(i, 0), (i, 1)\} : i \in Z_{25}\}$. First, we find a collection of 46 quadruples over Z_{25} by computer search, such that each triple of Z_{25} occurs in exactly two quadruples when developed on Z_{25}. Second, for each element of Z_{25}, we assign it a second coordinate, which is a linear function of a and b with $a, b \in Z_2$, such that for each triple $\{x, y, z\}$ in Z_{25}, the two occurrences are mapped into eight different triples in $\{x, y, z\} \times Z_2$. The 46 quadruples with $m \in Z_{25}$, $a \in Z_2$ and $b \in Z_2$ are listed below.

(m, a)	$(m+1, b)$	$(m+14, a)$	$(m+12, b)$
(m, a)	$(m+2, b)$	$(m+11, a+1)$	$(m+13, b+1)$
(m, a)	$(m+3, b)$	$(m+15, a)$	$(m+18, b)$
(m, a)	$(m+4, b)$	$(m+3, a)$	$(m+7, b)$
(m, a)	$(m+4, b)$	$(m+3, a+1)$	$(m+7, b+1)$
(m, a)	$(m+4, b)$	$(m+12, a+1)$	$(m+17, b+1)$
(m, a)	$(m+4, b)$	$(m+13, a+1)$	$(m+16, b+1)$
(m, a)	$(m+6, b)$	$(m+17, a+1)$	$(m+14, b+1)$

(m, a)	$(m+6, b)$	$(m+21, a+1)$	$(m+10, b+1)$
(m, a)	$(m+7, b)$	$(m+24, a)$	$(m+8, b)$
(m, a)	$(m+11, b)$	$(m+5, a)$	$(m+16, b)$
(m, a)	$(m+11, b)$	$(m+5, a+1)$	$(m+16, b+1)$
(m, a)	$(m+14, b)$	$(m+4, a)$	$(m+18, b)$
(m, a)	$(m+17, b)$	$(m+19, a)$	$(m+23, b)$
(m, a)	$(m+4, b)$	$(m+13, a)$	$(m+9, a+b)$
(m, a)	$(m+10, b)$	$(m+2, a)$	$(m+20, a+b)$
(m, a)	$(m+10, b)$	$(m+4, a)$	$(m+5, a+b)$
(m, a)	$(m+11, b)$	$(m+8, a)$	$(m+14, a+b)$
(m, a)	$(m+19, b)$	$(m+1, a)$	$(m+3, a+b)$
(m, a)	$(m+19, b)$	$(m+10, a)$	$(m+12, a+b)$
(m, a)	$(m+20, b)$	$(m+12, a)$	$(m+17, a+b)$
(m, a)	$(m+23, b)$	$(m+14, a)$	$(m+24, a+b)$
(m, a)	$(m+2, b)$	$(m+15, a+1)$	$(m+22, a+b)$
(m, a)	$(m+7, b)$	$(m+1, a+1)$	$(m+19, a+b)$
(m, a)	$(m+8, b)$	$(m+23, a+1)$	$(m+15, a+b)$
(m, a)	$(m+9, b)$	$(m+1, a+1)$	$(m+18, a+b)$
(m, a)	$(m+18, b)$	$(m+15, a+1)$	$(m+13, a+b)$
(m, a)	$(m+21, b)$	$(m+19, a+1)$	$(m+17, a+b)$
(m, a)	$(m+23, b)$	$(m+1, a+1)$	$(m+3, a+b)$
(m, a)	$(m+23, b)$	$(m+19, a+1)$	$(m+2, a+b)$
(m, a)	$(m+15, b)$	$(m+21, a)$	$(m+20, a+b+1)$
(m, a)	$(m+16, b)$	$(m+10, a)$	$(m+23, a+b+1)$
(m, a)	$(m+16, b)$	$(m+13, a)$	$(m+21, a+b+1)$
(m, a)	$(m+16, b)$	$(m+14, a)$	$(m+15, a+b+1)$
(m, a)	$(m+17, b)$	$(m+13, a)$	$(m+22, a+b+1)$
(m, a)	$(m+17, b)$	$(m+2, a)$	$(m+7, a+b+1)$
(m, a)	$(m+22, b)$	$(m+8, a)$	$(m+19, a+b+1)$
(m, a)	$(m+23, b)$	$(m+1, a)$	$(m+7, a+b+1)$
(m, a)	$(m+7, b)$	$(m+21, a+1)$	$(m+14, a+b+1)$
(m, a)	$(m+7, b)$	$(m+24, a+1)$	$(m+16, a+b+1)$
(m, a)	$(m+13, b)$	$(m+10, a+1)$	$(m+19, a+b+1)$
(m, a)	$(m+15, b)$	$(m+14, a+1)$	$(m+24, a+b+1)$
(m, a)	$(m+16, b)$	$(m+10, a+1)$	$(m+22, a+b+1)$
(m, a)	$(m+20, b)$	$(m+2, a+1)$	$(m+7, a+b+1)$
(m, a)	$(m+24, b)$	$(m+4, a+1)$	$(m+5, a+b+1)$
(m, a)	$(m+24, b)$	$(m+11, a+1)$	$(m+12, a+b+1)$

□

The following lemma is useful for us to unify the proofs following-up, which also provides another proof for the existence of $S(3, \{4, 6\}, v)$ with some small initial ingredients.

Lemma 3.3 *For each integer* $n \geq 3$*, there exists a* $CS(3, \{4, 6\}, 2n+2)$ *of type* $(2^{n-2\epsilon}4^{\epsilon} : 2)$ *with* $\epsilon = 0$ *or* 1.

Proof For each integer $n \geq 3$, it is sufficient to prove that there exists an $S(3, \{4, 6\}, 2n+2)$ $(X, \mathcal{A})$ such that the design has two particular points $\{x, y\} \subset X$ with at most one block of size 6 containing both of them.

For $n = 3, 4$, the conclusion is true since an SQS$(2n+2)$ exists. For $n = 5$, there exists an $S(3, \{4, 6\}, 12)$ with two disjoint blocks of size 6 partitioning the point set, which can be obtained from a GDD$(3, \{4, 6\}, 12)$ of type 2^6 [5, Lemma 1].

For $n > 5$, assume that the conclusion is true for each i, $3 < i < n$. The proof proceeds by induction.

Firstly, suppose that there exists an $S(3, \{4, 6\}, n+1)$ $(X, \mathcal{A})$ with two particular points $\{x, y\} \subset X$, such that there is at most one block of size 6 containing $\{x, y\}$. Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a one-factorization of the complete graph on X. Construct an $S(3, \{4, 6\}, 2n + 2)$ on $X \times \{0, 1\}$ with block set $\mathcal{B} = (\mathcal{A} \times \{0, 1\}) \cup \mathcal{C}$, where $\mathcal{C} = \{\{(a, 0), (b, 0), (c, 1), (d, 1)\} : \{a, b\} \in F_i, \{c, d\} \in F_i, 1 \leq i \leq n\}$. It is not difficult to check that there is at most one block of size 6 in $\mathcal{B}$ containing $\{(x, 0), (y, 0)\}$.

Secondly, suppose that there exists an $S(3, \{4, 6\}, n+2)$ $(X, \mathcal{A})$ with two particular points $\{x, y\} \subset X$, such that there is at most one block of size 6 containing $\{x, y\}$. Take a point $\infty \in X \backslash \{x, y\}$ and let $X' = (X \backslash \{\infty\}) \times \{0, 1\}$. For each block $A \in \mathcal{A}$ containing ∞, construct a $CS(3, \{4, 6\}, 2|A| - 2)$ of type $(2^{|A|-1} : 0)$ on $(A \backslash \{\infty\}) \times \{0, 1\}$. For each block A not containing ∞, construct a $GDD(3, \{4, 6\}, 2|A|)$ of type $2^{|A|}$ on $A \times \{0, 1\}$. When $|A| = 6$, let $A \times \{0\}$ and $A \times \{1\}$ be the two special blocks of size 6 of the input $GDD(3, \{4, 6\}, 12)$ of type 2^6. By Theorem 2.1, we get a $CS(3, \{4, 6\}, 2n + 2)$ of type $(2^{n+1} : 0)$, which is actually an $S(3, \{4, 6\}, 2n + 2)$ on X'. Here, the input $CS(3, \{4, 6\}, 6)$ of type $(2^3 : 0)$ contains only one block of size 6. The input $CS(3, \{4, 6\}, 10)$ of type $(2^5 : 0)$ is actually an SQS(10) which contains only blocks of size 4. Take the two points $\{(x, 0), (y, 1)\}$ into consideration. If $\{\infty, x, y\}$ determines a block of size 6 in $\mathcal{A}$, then there is no block of size 6 containing $\{(x, 0), (y, 1)\}$. If $\{\infty, x, y\}$ determines a block of size 4 in $\mathcal{A}$, then there is only one block of size 6 containing $\{(x, 0), (y, 1)\}$. □

Remark For $n = 3, 4, 5$, it is easy to check that each of the $S(3, \{4, 6\}, 2n + 2)$'s has blocks of size four not containing the particular pair $\{x, y\}$. So does the $S(3, \{4, 6\}, 2n + 2)$ with $n \geq 3$ by induction as in Lemma 3.3. Hence, there is at least one block of size four in the resultant $CS(3, \{4, 6\}, 2n + 2)$ for all $n \geq 3$.

Lemma 3.4 *There exists an* $H(2^n)$ *for all* $n \equiv 5 \pmod 6$, $n \geq 11$ *and an* $IH(2^n : 2^4)$ *for all* $n \equiv 5 \pmod 6$, $n \geq 17$.

Proof For $n = 11$, an $H(2^{11})$ exists by Lemma 3.1. For $n = 17$, applying Corollary 2.7 with $(n, s) = (7, 2)$ and an $H(2^7)$ from Lemma 3.1, we obtain an $IH(2^{17} : 2^4)$ and an $H(2^{17})$.

For each $n = 6m + 5$, $m \geq 3$, there exists a $CS(3, \{4, 6\}, 2m + 2)$ of type $(2^{m-2\epsilon} 4^{\epsilon} : 2)$ with $\epsilon = 0$ or 1 by Lemma 3.3. By the Remark after Lemma 3.3, there exists a block of size four, say B, in the block set of the $CS(3, \{4, 6\}, 2m + 2)$. Take any point from the two stem points and define it as the infinite point, which is outside of B. Then apply Lemma 2.2 with an $HF_2(3^{k-1} : 2)$ and an $H(6^k)$ for $k \in \{4, 6\}$ to obtain an $HF_2(6^{m-2\epsilon} 12^{\epsilon} : 5)$ with a subdesign $H(2^4)$. Applying Lemma 2.3 with an $IH(2^{11} : 2^5)$, an $H(2^{11})$ or an $H(2^{17})$, we get an $H(2^{6m+5})$ with a subdesign $H(2^4)$. Here, the input $HF_2(3^{k-1} : 2)$ comes from Theorem 2.8 or [17, Lemma 6.12], the input $H(2^{17})$ is constructed above, and the other ingredients are from Lemma 3.1. □

Lemma 3.5 *There exists an* $H(2^n)$ *for all* $n \equiv 7, 13 \pmod{18}$ *and* $n \geq 7$.

Proof For each $n = 18k + 7$ and $k \geq 2$, we obtain an $IH(2^n : 2^4)$ by applying Corollary 2.7 with an $IH(2^{6k+5} : 2^4)$ from Lemma 3.4. Applying Lemma 2.3 with an $H(2^4)$, we obtain an $H(2^n)$. For $n = 7, 25$, the required designs exist by Lemmas 3.1 and 3.2.

For each $n = 18k + 13$ and $k \geq 1$, there is an $H(2^n)$ by applying Corollary 2.7 with an $IH(2^{6k+5} : 2^1)$ from Lemma 3.4. For $n = 13$, the required design exists by Lemma 3.1. □

Lemma 3.6 *There exists an* $H(2^n)$ *for all* $n \equiv 1 \pmod{18}$.

Proof For each $n = 18k + 1$ and $k \geq 1$, the proof proceeds by induction. For $k = 1$, an $H(2^{19})$ exists by applying Corollary 2.9 with an $IH(2^7 : 2^1)$. When $k > 1$, suppose that there exists an $H(2^{18i+1})$ for each $i < k$. By Lemma 3.5, we have that an $H(2^{6j+1})$ exists for all $j < 3k$. Applying Corollary 2.9 with an $IH(2^{6k+1} : 2^1)$, we get an $H(2^{18k+1})$. □

Theorem 3.7 *There exists an* $H(2^n)$ *for all* $n \equiv 1, 2 \pmod 3$ *and* $n \neq 5$.

Proof Combining Lemmas 3.4–3.6, we obtain an $\mathrm{H}(2^n)$ for each $n \equiv 1, 5 \pmod 6$ and $n \neq 5$. By Theorem 2.10, we obtain an $\mathrm{H}(2^m)$ for each $m \equiv 2, 4 \pmod 6$ and $m \neq 10$. An $\mathrm{H}(2^{10})$ can be obtained by applying Corollary 2.9 with an $\mathrm{IH}(2^4 : 2^1)$. □

As a consequence of Theorem 3.7, we have the following corollary.

Corollary 3.8 *There exists an* $SQS(v)$ *for all* $v \equiv 2, 4 \pmod 6$.

Proof The existence of $\mathrm{SQS}(v)$ with small orders of $v = 4, 8, 10$ was mentioned in Sect. 1. Combining every two groups of an $\mathrm{H}(2^n)$ to form a quadruple as a new block, we get an $\mathrm{SQS}(2n)$ for each $n \equiv 1, 2 \pmod 3$ and $n \geq 7$. □

4 Concluding remarks

In this paper, we gave a new existence proof for Steiner quadruple systems by reestablishing the existence of H-designs of type 2^n based on the theory of candelabra systems and H-frames. This new approach has been proved to be quite effective to deal with the existence problems for optimal constant weight covering codes and nonuniform H-designs of types $2^n u^1$ with $u = 6, 8$ [17]. We believe that the theory of candelabra systems and H-frames will be proved useful for a complete solution of the general existence problem on H-designs of type $g^n u^1$.

Acknowledgments The authors express their gratitude to the anonymous reviewer for the detailed and constructive comments which are very helpful to the improvement of the technical presentation of this paper. The authors also thank Professor Zhu Lie for his helpful suggestions on this problem. Research of the second author was supported by the National Outstanding Youth Science Foundation of China under Grant no. 10825103, National Natural Science Foundation of China under Grant no. 61171198, Specialized Research Fund for the Doctoral Program of Higher Education, and Zhejiang Provincial Natural Science Foundation of China under Grant no. D7080064.

References

1. Barrau J.A.: On the combinatory problem of Steiner. Proc. Sec. Sci. Kon Akad. Wetensch. Amst. **11**, 352–360 (1908).
2. Carmichael R.D.: Introduction to the Theory of Groups of Finite Order. Ginn, Boston (1937); reprinted by Dover, New York (1956).
3. Chouinard L.G. II, Kramer E.S., Kreher D.L.: Graphical t-wise balanced designs. Discret. Math. **46**, 227–240 (1983).
4. Hanani H.: On quadruple systems. Can. J. Math. **12**, 145–157 (1960).
5. Hanani H.: On some tactical configurations. Can. J. Math. **15**, 702–722 (1963).
6. Hartman A.: Tripling quadruple systems. Ars Combin. **10**, 255–309 (1980).
7. Hartman A.: A general recursive construction for quadruple systems. J. Combin. Theory Ser. A. **33**, 121–134 (1982).
8. Hartman A.: The fundamental constructions for 3-designs. Discrete Math. **124**, 107–132 (1994).
9. Hartman A., Mills W.H., Mullin R.C.: Covering triples by quadruple: an asymptotic solution. J. Combin. Theory Ser. A **41**, 117–138 (1986).
10. Hartman A., Phelps K.T.: Steiner quadruple systems. In: Dinitz, J.H., Stinson, D.R. (eds.) Contemporary Design Theory: A Collection of Surveys. Wiley, New York, pp. 117–138.
11. Ji L.: An improvement on H design. J. Combin. Des. **17**, 25–35 (2009).
12. Kirman T.P.: On a problem in combinations. Camb. Dublin Math. J. **2**, 191–204 (1847).
13. Lenz H.: Tripling Steiner quadruple systems. Ars Combin. **20**, 193–202 (1985).

14. Mills W.H.: On the covering of triples by quadruples. Congr. Numer. **10**, 563–581 (1974).
15. Mills W.H.: On the existence of H designs. Congr. Numer. **79**, 129–141 (1990).
16. Stern G., Lenz H.: Steiner triple systems with given subspaces, another proof of the Doyen-Wilson Theorem. Boll. Un. Mat. Ital. (5) **17-A**, 109–114 (1980).
17. Zhang X., Zhang H., Ge G.: Optimal constant weight covering codes and nonuniform group divisible 3-designs with block size four. Des. Codes Cryptogr. **62**, 143–160 (2012).